Integer Gradient Descent

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Models of Computation

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- **HALTS**: If $x_{n+1} = x_n$

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- Finally, we can simplify the dynamics further by noting that we can choose $F(x) = (-\tilde{F}(x) + \frac{1}{2}x^2)$, leading to $x_{n+1} = x_n - \frac{d}{dx} F(x)\Big|_{x=x_n} = \frac{d}{dx}\tilde{F}(x)\Big|_{x=x_n}$. Leading to the simple **Dynamics**:

$$x_{n+1} = f(x_n) \tag{7}$$

$$\text{with } f(x_n) = \frac{d}{dx}\tilde{F}(x)\Big|_{x=x_n} \tag{8}$$

Example: Collatz Sequence

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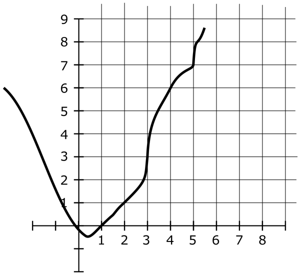

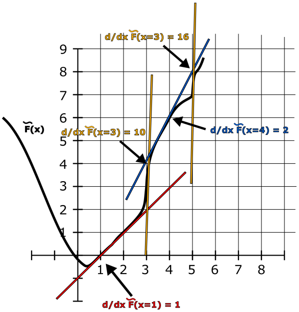

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- Instead: $|3x - 3y| > q|x - y|$ if x uneven and y uneven.

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- The presented model is similar to FRACTRANS. A FRACTRAN program consists of an ordered list of fractions $\{f_1, f_2, ..., f_n\}$, $f_i \in \mathbb{Q}$ and the input $x_n \in \mathbb{N}$ is multiplied with these fractions. The first multiplication that yields an integer replaces x_n with $x_{n+1} = f_i \cdot x_n$.

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- Our model is similar in that we can interpret the FRACTRAN fractions $f_1, ..., f_n$ also as a set of linear functions $f^i(x) : \mathbb{R} \rightarrow \mathbb{R}$ that are evaluated in sequence: $x_{n+1} = f^i(x_n) = f_i \cdot x_n$ if $f^i(x_n) \in \mathbb{N}$. In our case the tangents of the function $\tilde{F}$: $\frac{d}{dx}\tilde{F}$ take the place of f^i.

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  - $\Rightarrow x_{10} = \frac{3}{2 \cdot 7 \cdot 11 \cdot 13} 2^4 3^{-1} 7^1 11^7 13^1 = 2^3 3^0 7^0 11^6 13^0$ (HALT)

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